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Abelian decomposition of $SO(2N)$ Yang-Mills theory

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Abstract. Faddeev and Niemi have proposed a decomposition of $SU(N)$ Yang–Mills theory in terms of new variables, appropriate for describing the theory in the infrared limit. We extend this method to $SO(2N)$ Yang–Mills theory. We find that the $SO(2N)$ connection decomposes according to irreducible representations of $SO(N)$. The low-energy limit of the decomposed theory is expected to describe soliton-like configurations with nontrivial topological numbers. How the method of decomposition generalizes for $SO(2N+1)$ Yang–Mills theory is also discussed.

The mechanism of color confinement in Yang–Mills theory is known to be one of most difficult problems in theoretical physics. A qualitative explanation of this problem is provided by monopole condensation, which causes the confinement of color through the dual Meissner effect [1,2]. It is conjectured that an Abelian projection of the Yang–Mills theory to its maximal Abelian subgroup is responsible for the dynamics of the dual Meissner effect. That is to say, in the infrared limit the degrees of freedom of a non-Abelian theory are dominated by those of its maximal Abelian subgroup [2]. However, a quantitative understanding on how the monopole condenses in the low-energy limit, starting from the fundamental Yang–Mills theory, is still absent and awaits to be further explored.

Recently, Faddeev and Niemi have proposed an Abelian decomposition of the four-dimensional $SU(2)$ Yang–Mills connection A^a_μ [3]. The decomposed theory, which is appropriate for describing the Yang–Mills theory in its infrared limit, involves an Abelian gauge field C_μ, a complex scalar field $\phi = \rho + \mathrm{i}\sigma$, and a three component unit vector field n^a. It is an on-shell decomposition because the variations of the decomposed theory with respect to the fields (C_μ, ϕ, n^a) reproduce the equations of motion of the original $SU(2)$ Yang–Mills theory. It is shown, on the one hand, that if the fields (C_μ, ϕ) are properly integrated out, the resulting theory supports a string-like knotted solution, which describes at large distance the dynamics of extended, massive knot-like solitons [4]. These solitonic configurations can be regarded as the natural candidates for describing glueballs. On the other hand, if the vector field n^a is averaged over first, the multiplet (C_μ, ϕ) transforms as the fields in the Abelian Higgs model.

The method of an Abelian decomposition based on $SU(2)$ Yang–Mills theory is readily generalized to the general case of four-dimensional $SU(N)$ Yang–Mills theory [5, 6]. It is found that the $SU(N)$ connection decomposes according to irreducible representations of $SO(N-1)$ and that the low-energy limit of the decomposed theory may describe stable, soliton-like configurations with nontrivial topological numbers [6]. The first principle derivation on the effective action that describes the Yang–Mills theories in the infrared limit can be found in [7–9].

In this letter, we extend the Abelian decomposition of the four-dimensional $SU(N)$ Yang–Mills connection to the case of $SO(2N)$ gauge theory. We shall construct the N mutually orthogonal Lie-algebra valued vector fields m_i with unit length so that they describe $2N(N-1)$ independent variables. Then we use the fields m_i to construct several special $SO(2N)$ covariant one-forms, that are orthogonal to m_i and determine a basis of roots in $SO(2N)$. Consequently, the combination of the fields m_i and the covariant one-forms constructed from m_i yields a complete set of basis states for the $SO(2N)$ Lie algebra. All together, they will be used to decompose the generic $SO(2N)$ connections. In the concluding part of this letter, we discuss the generalization of the Abelian decomposition for $SO(2N+1)$ Yang–Mills theory. It is straightforward, provided the decomposition of $SO(2N)$ theory has been established.

The $SO(2N)$ Lie group is rank N and its Lie algebra has $N(2N-1)$ generators. We denote them by $T_{a,b}$ with the antisymmetric property, i.e., $T_{a,b} = -T_{b,a}$ for $a, b = 1$ to $2N$. The generators are chosen in the defining representation as follows:

$$[T_{a,b}]_{c,d} = -\frac{\mathrm{i}}{\sqrt{2}} \left(\delta_{ac}\delta_{bd} - \delta_{bc}\delta_{ad}\right). \tag{1}$$

Here, we have normalized the generators such that $\mathrm{Tr}(T_{a,b} T_{c,d}) = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}$. The commutation relations among the generators (1) are easily obtained:

$$[T_{a,b}, T_{c,d}] = \frac{\mathrm{i}}{\sqrt{2}} \left(\delta_{ac}T_{b,d} + \delta_{bd}T_{a,c} - \delta_{ad}T_{b,c} + \delta_{bc}T_{a,d}\right). \tag{2}$$

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This is the form for the rotational generators in a $2N$-dimensional real vector space.

We designate the basis of the commuting generators in the Cartan subalgebra

$$H_{2i-1,2i} = T_{2i-1,2i}, \tag{3}$$

where $i = 1$ to N. Note that $[H_{2i-1,2i}, H_{2j-1,2j}] = 0$. In terms of the generators $T_{a,b}$, a generic Lie-algebra element v has the expansion

$$v = \frac{1}{2} v^{a,b} T_{a,b}$$

The factor of one half is needed to avoid double counting in the contraction summation.

Following [6], we now conjugate the elements of the Cartan subalgebra $H_{2i-1,2i}$ (3) by a generic element $g \in SO(2N)$. This gives N Lie-algebra valued vector fields. They are

$$m_i = g H_{2i-1,2i} g^{-1} = \frac{1}{2} m_i^{a,b} T_{a,b}. \tag{4}$$

Note that the fields m_i remain invariant if g transforms by a right diagonal factor $g \to gh$, with h belonging to the maximal Abelian subgroup of $SO(2N)$. In this way, m_i produce an over-determined set of coordinates on the orbit $SO(2N)/U(1)^N$ and depend on only $2N(N-1)$ independent variables. In addition, they are orthonormal:

$$(m_i, m_j) \equiv \mathrm{Tr}\,(m_i m_j) = \frac{1}{2} m_i^{a,b} m_j^{a,b} = \delta_{ij}. \tag{5}$$

Using (4), it is straightforward to verify that

$$[m_i, m_j] = 0, \tag{6}$$

$$\mathrm{Tr}\,(m_i \mathrm{d} m_j) = (m_i, \mathrm{d} m_j) = 0, \tag{7}$$

where

$$\mathrm{d} m_j = \partial_\mu m_j \mathrm{d} x^\mu.$$

Next, we proceed to consider an arbitrary Lie-algebra element v under an infinitesimal adjoint action on the fields m_i. We define this action by

$$\delta^i v = [v, m_i]\,. \tag{8}$$

Applying the action δ^i twice and summing over the index i, we obtain a projection operator to a subspace which is orthogonal to the maximal torus and is spanned by the Lie-algebra valued fields m_i,

$$(\delta^i)^2 v = v - m_i (m_i, v). \tag{9}$$

Note that the subspace in which (9) projects corresponds to the space $SO(2N)/U(1)^N$, i.e., the roots of $SO(2N)$. To derive (9), we make use of (2), (5), and this equation:

$$\sum_i \left[[\tilde{v}, H_{2i-1,2i}], H_{2i-1,2i} \right] = \tilde{v} - \tilde{v}^{2i-1,2i} H_{2i-1,2i},$$

where $\tilde{v} = g^{-1} v g$.

Having presented the basic formulas needed, we hope to generalize the method of Abelian decomposition for $SO(2N)$ Yang–Mills theory. Introducing the matrix notation for the $SO(2N)$ connection one-form

$$A = A_\mu \mathrm{d} x^\mu = \frac{1}{2} A_\mu^{a,b} T_{a,b} \mathrm{d} x^\mu, \tag{10}$$

we parameterize this connection one-form A to obtain the following expression:

$$A = C^i m_i + \frac{1}{i} [\mathrm{d} m_i, m_i] + (\text{covariant part}). \tag{11}$$

The combination of the first two terms on the right-hand-side of (11) is the so-called Cho connection, which was first introduced as a consistent truncation of the full four-dimensional connection [10]. It can be shown that, under N independent gauge transformations generated by the Lie-algebra elements $\alpha^i m_i$, the Cho connection retains the full non-Abelian gauge degrees of freedom, while the one-forms C^i transform as $U(1)$ connections, $C^i \to C^i + \mathrm{d}\alpha^i$. Hence, the remaining part on the right-hand side of (11) (covariant part) must transform covariantly under gauge transformations and by construction must be orthogonal to the fields m_i.

Because the decomposition method introduced by Faddeev and Niemi is on-shell complete, the number of field multiplets that appear in the decomposed connection (11) has to be equal to that of physically relevant field degrees of freedom carried by the original $SO(2N)$ connection. It is known that the $SO(2N)$ Yang–Mills connection (10) contains $2N(2N-1)$ physical components. On the contrary, the Cho connection in (11) introduces N $U(1)$ connections C^i and N vector fields m_i. The former contributes $2N$ physical degrees of freedom, while the latter describes $2N(N-1)$ independent variables. Adding both contributions gives $2N^2$. As a result, the difference in degrees of freedom between both connections is

$$2N(2N-1) - 2N^2 = 2N(N-1). \tag{12}$$

This is the number of independent variables held exactly by the (covariant part) of (11). So, the space of what we called (covariant part) is $2N(N-1)$-dimensional. Moreover, according to the definition the fields appearing in the (covariant part) of (11) are orthogonal to the fields m_i. We thus deduce that the space of the (covariant part) coincides with the subspace to which the operator (9) projects, the orbit $SO(2N)/U(1)^N$.

In the following paragraphs, we shall use the fields m_i to construct certain special Lie-algebra valued one-forms which determine the local basis of the (covariant part) space. What are these Lie-algebra valued one-forms? They can be gotten by repeatedly using the adjoint action (8). For instance, we first learn from (7) that the Lie-algebra valued one-forms $\mathrm{d} m_i$ are orthogonal to m_k. Let us denote $\mathrm{d} m_i$ by iX_i for the purpose of later convenience and identify the one-forms X_i as one subset of the basis states of the (covariant part) space. Next, we apply the adjoint action (8) on X_i to obtain other one-forms Z_{ij},

$$Z_{ij} \equiv \delta^j X_i = [X_i, m_j]\,. \tag{13}$$

It is not difficult to see that, by utilizing (6), the one-forms Z_{ij} are orthogonal to m_k, too. Hence, the one-forms Z_{ij} can also be used to parameterize the basis of the (covariant part) space. Consequently, we continue to find the remaining one-forms that span the (covariant part) space by recurrently applying the adjoint action (8) on the latest generated one-forms. After a little manipulation, we have

$$\begin{aligned}\delta^k Z_{ij} &= \frac{1}{2}\delta_{ij}\left[\frac{1}{2}\left(\delta_{ik}+\delta_{jk}\right)X_k + V_{ik} + V_{jk}\right] \\ &\quad + \delta_{ik}V_{kj} + \delta_{jk}V_{ki}, \end{aligned}\tag{14}$$

$$\delta^k V_{ij} = \frac{1}{2}\delta_{ik}Z_{kj} - \delta_{jk}U_{ki}, \tag{15}$$

$$\begin{aligned}\delta^k U_{ij} &= \frac{1}{4}\delta_{ij}\left[\frac{1}{2}\left(\delta_{ik}+\delta_{jk}\right)X_k + V_{ik} + V_{jk}\right] \\ &\quad - \frac{1}{2}\left(\delta_{ik}V_{jk} + \delta_{jk}V_{ik}\right). \end{aligned}\tag{16}$$

It turns out that we get four subsets of Lie-algebra valued one-forms $(X_i, Z_{ij}, V_{ij}, U_{ij})$ in total, which form a closed algebra under the adjoint action (8). The details of these one-forms are separately given in the appendix.

The one-forms $(X_i, Z_{ij}, V_{ij}, U_{ij})$ possess definite properties under $SO(N)$ symmetries, for N specifies the rank of $SO(2N)$. See the appendix for details. For example, the one-forms X_i yield the $SO(N)$ vector representation, V_{ij} the $SO(N)$ rank-two tensor representation, and Z_{ij} and U_{ij} the $SO(N)$ symmetric tensor representations. However, not all of the components in the one-forms V_{ij} and U_{ij} are independent. It is shown in the appendix that the rank-two tensor V_{ij} satisfies two sets of constraints:

$$\sum_i V_{ij} = \frac{1}{2}X_j,$$

and

$$V_{ii} = 0$$

(no summation), and that the symmetric tensor U_{ij} obeys also two sets of constraints:

$$\sum_i U_{ij} = 0,$$

and

$$U_{ii} = \frac{1}{2}Z_{ii}$$

(no summation).

After all this, we are enabled to count the number of independent components possessed by each of the one-forms X_i, Z_{ij}, V_{ij}, and U_{ij}. The dimension of the vector X_i is N and the dimension of the symmetric tensor Z_{ij} is $(1/2N)(N+1)$. Analogously, after taking the constraint equations into account, the dimension of the second rank tensor V_{ij} is $N^2 - 2N$ and the dimension of the other symmetric tensor U_{ij} is $(1/2)N(N+1) - 2N$. The sum of these four numbers is $2N(N-1)$, which as expected coincides with the dimension of the space $SO(2N)/U(1)^N$.

As a result, $(m_i, X_i, Z_{ij}, V_{ij}, U_{ij})$ yields a complete set of basis states for the $SO(2N)$ Lie algebra, and can be used to decompose generic $SO(2N)$ connections. To complete the decomposition, we need appropriate dual variables that appear as coefficients. We observe that the Yang–Mills connection A in (10) is an $SO(2N)$ Lie-algebra valued one-form and transforms in the scalar representation of the $SO(N)$ group. Accordingly, the variables that are dual to the one-forms $(X_i, Z_{ij}, V_{ij}, U_{ij})$ are undoubtedly zero-forms. Let us denote them by $(\phi^i, \psi^{ij}, \sigma^{ij}, \rho^{ij})$, respectively. These dual variables must transform in the same $SO(N)$ representations as the associated one-forms in order to form invariant combinations[1].

We therefore conclude that the following decomposition of the four-dimensional $SO(2N)$ connection contains the correct number of independent variables, which are appropriate for describing the theory in the low-energy limit,

$$A = C^i m_i + \phi^i X_i + \left(\delta_{ij} + \psi^{ij}\right) Z_{ij} + \sigma^{ij} V_{ij} + \rho^{ij} U_{ij}. \tag{17}$$

According to the discussion of the appendix, (17) can also be expressed in a gauge equivalent form

$$\begin{aligned}\tilde{A} &= \left(C^i - \frac{1}{i}R^{2i-1,2i}\right) H_{2i-1,2i} + \phi^i x_i + \psi^{ij} z_{ij} \\ &\quad + \sigma^{ij} v_{ij} + \rho^{ij} u_{ij}, \end{aligned}\tag{18}$$

where

$$(x_i, z_{ij}, v_{ij}, u_{ij}) = g^{-1}\left(X_i, Z_{ij}, V_{ij}, U_{ij}\right) g.$$

The Wilsonian renormalization group argument suggests that, in terms of the field variables of the decomposed connection (17), the infrared $SO(2N)$ Yang–Mills theory takes the form

$$S(m_i) = \int \mathrm{d}^4 x \left[(\partial_\mu m_i)^2 + \frac{1}{e_i^2}\left([\partial_\mu m_i, \partial_\nu m_i]\right)^2\right]. \tag{19}$$

The action (19) is in the same universality class as that obtained in [6] and is expected to describe stable, soliton-like configurations with nontrivial topological numbers, since

$$\pi_3(SO(4)/U(1)^2) = Z + Z$$

and

$$\pi_3(SO(2N)/U(1)^N) = Z$$

for $N \geq 3$. It is interesting to investigate the detailed structures of the action.

In conclusion, we briefly summarize how the method of Abelian decomposition generalizes for $SO(2N+1)$ Yang–Mills connection. The $SO(2N+1)$ Lie algebra is rank

[1] The $U(1)$ connection one-forms C^i in (11) are the dual variables to the zero-forms m_i. Thus, the $SO(N)$ group acts on the combination $C^i m_i$ trivially

N and has $N(2N+1)$ generators. As usual, we denote the generators by $T_{a,b}$ for $a, b = 1$ to $N+1$. The N Lie-algebra valued vector fields m_i are constructed similar to (4) except that they depend on $2N^2$ independent variables. The $SO(2N+1)$ connection one-form can still be parameterized by (11), but the dimension of the space of the (covariant part) is $2N^2$. Needless to say, the (covariant part) space is identical to the space $SO(2N+1)/U(1)^N$. The set of Lie-algebra one-forms which determine the local basis of the space $SO(2N+1)/U(1)^N$, is found to be $(X_i, Z_{ij}, V_{ij}, U_{ij})$. At this time, the one-forms V_{ij} fulfill one set of constraints instead of two. It is

$$V_{ii} = 0$$

(no summation). As regards the one-forms U_{ij}, the set of constraint satisfied by them is

$$U_{ii} = \frac{1}{2} Z_{ii}$$

(no summation). It turns out that the number of independent variables carried by the set of one-forms ($X_i, Z_{ij}, V_{ij}, U_{ij}$) is $2N^2$, matching the dimension of the (covariant part) space. Therefore, the expression (17) for the connection one-form is perfectly applicable to the decomposition of $SO(2N+1)$ Yang–Mills theory.

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Appendix

We explicitly give the Lie-algebra valued one-forms X_i, Z_{ij}, V_{ij}, and U_{ij}, expanded in terms of the generators $T_{a,b}$, (1). In (17), these one-forms are used to parameterize the local basis of the orbit $SO(2N)/U(1)^N$.

To begin with, we introduce the Maurer–Cartan one-forms

$$L = \mathrm{d}g g^{-1} \quad \text{and} \quad R = g^{-1}\mathrm{d}g, \tag{20}$$

then use (4) and (20) to rewrite

$$\mathrm{d}m_i = [L, m_i] = g\,[R, H_{2i-1,2i}]\, g^{-1}, \tag{21}$$

$$\begin{aligned}[\mathrm{d}m_i, m_i] &= g\left(R - R^{2i-1,2i} H_{2i-1,2i}\right) g^{-1} \\ &= L - g\left(R^{2i-1,2i} H_{2i-1,2i}\right) g^{-1}. \end{aligned}\tag{22}$$

Because the (covariant part) of (11) transforms covariantly under gauge transformation, we further represent the connection one-form (11) in the form of a manifestly gauge equivalent expression:

$$\begin{aligned} A = g&\left[\left(C^i - \frac{1}{i} R^{2i-1,2i}\right) H_{2i-1,2i} + (\text{c.p.})\right] g^{-1} \\ &+ \frac{1}{i}\mathrm{d}g g^{-1}, \end{aligned}\tag{23}$$

where

$$(\text{c.p.}) = g^{-1}(\text{covariant part})g.$$

Similar to what we have shown on the local basis of the (covariant part) space in (11), the space of (c.p.) in (23) is likewise spanned by four Lie-algebra one-forms ($x_i, z_{ij}, v_{ij}, u_{ij}$). They are related to the one-forms ($X_i, Z_{ij}, V_{ij}, U_{ij}$) of the (covariant part) as follows.

The $SO(N)$ vector one-form x_i is defined by

$$x_i = g^{-1} X_i g,$$

with

$$x_i = \frac{1}{\sqrt{2}}\left(R^{2i,a} T_{2i-1,a} - R^{2i-1,a} T_{2i,a}\right). \tag{24}$$

Similarly, after introducing the set of identities

$$(z_{ij}, v_{ij}, u_{ij}) = g^{-1}\left(Z_{ij}, V_{ij}, U_{ij}\right) g,$$

we have

$$\begin{aligned} z_{ij} = \frac{1}{2i}[&\delta_{ij}\left(R^{2i,a} T_{2j,a} + R^{2i-1,a} T_{2j-1,a}\right) \\ &+ R^{2i,2j-1} T_{2i-1,2j} + R^{2i-1,2j} T_{2i,2j-1} \\ &- R^{2i,2j} T_{2i-1,2j-1} - R^{2i-1,2j-1} T_{2i,2j}], \end{aligned}\tag{25}$$

$$\begin{aligned} v_{ij} = \frac{1}{2\sqrt{2}}[&R^{2i,2j} T_{2i,2j-1} + R^{2i-1,2j} T_{2i-1,2j-1} \\ &- R^{2i,2j-1} T_{2i,2j} - R^{2i-1,2j-1} T_{2i-1,2j}], \end{aligned}\tag{26}$$

$$\begin{aligned} u_{ij} = \frac{1}{4i}[&\delta_{ij}\left(R^{2i,a} T_{2j,a} + R^{2i-1,a} T_{2j-1,a}\right) \\ &- R^{2i,2j} T_{2i,2j} - R^{2i,2j-1} T_{2i,2j-1} - R^{2i-1,2j} T_{2i-1,2j} \\ &- R^{2i-1,2j-1} T_{2i-1,2j-1}]. \end{aligned}\tag{27}$$

It is apparent from (26) and (27) that not all the components of v_{ij} and u_{ij} are independent. The rank-two tensor v_{ij} (26) satisfies

$$\sum_i v_{ij} = \frac{1}{2} x_j, \tag{28}$$

$$v_{ii} = 0 \tag{29}$$

(no summation). In the same vein, in (27) we find two sets of constraints fulfilled by u_{ij},

$$\sum_i u_{ij} = 0, \tag{30}$$

$$u_{ii} = \frac{1}{2} z_{ii} \tag{31}$$

(no summation).

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